11-6 Recursion and Special Sequences

Main Ideas

- Recognize and use special sequences.
- Iterate functions.

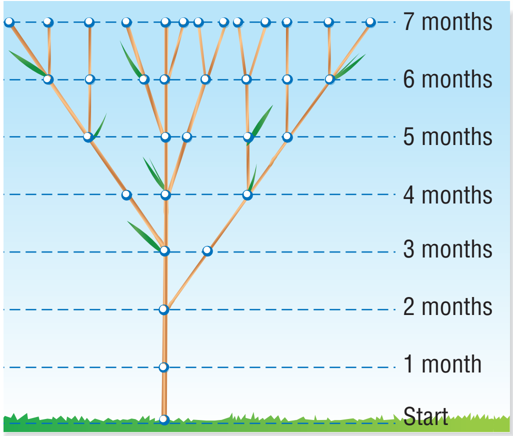
New Vocabulary

Fibonacci sequence
recursive formula
iteration

GET READY for the Lesson

A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Month	1	2	3	4	5
Shoots	1	1	2	3	5



Special Sequences Notice that each term in the sequence is the sum of the two previous terms. For example, 8 = 3 + 5 and 13 = 5 + 8. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	a_3	$a_1 + a_2$	$1 + 1 = 2$
fourth term	a_4	$a_2 + a_3$	$1 + 2 = 3$
⋮	⋮	⋮	
nth term	a_n	$a_{n-2} + a_{n-1}$	

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms.

EXAMPLE 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.

$a_{n+1} = 3a_n - 2$ — Recursive formula

$a_{1+1} = 3a_1 - 2$ — $n = 1$

$a_2 = 3(4) - 2$ or 10 — $a_1 = 4$

$a_{2+1} = 3a_2 - 2$ — $n = 2$

$a_3 = 3(10) - 2$ or 28 — $a_2 = 10$

$a_{3+1} = 3a_3 - 2$ — $n = 3$

$a_4 = 3(28) - 2$ or 82 — $a_3 = 28$

$a_{4+1} = 3a_4 - 2$ — $n = 4$

$a_5 = 3(82) - 2$ or 244 — $a_4 = 82$

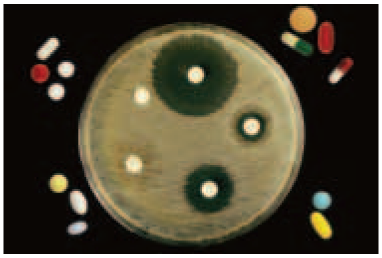
The first five terms of the sequence are 4, 10, 28, 82, and 244.

CHECK Your Progress

1. Find the first five terms of the sequence in which $a_1 = -1$ and $a_{n+1} = 2a_n + 4$, $n \geq 1$.

Real-World EXAMPLE Find and Use a Recursive Formula

2 MEDICAL RESEARCH A pharmaceutical company is experimenting with a new drug. An experiment begins with 1.0×10^9 bacteria. A dose of the drug that is administered every four hours can kill 4.0×10^8 bacteria. Between doses of the drug, the number of bacteria increases by 50%.



Real-World Link

In 1928, Alexander Fleming found that penicillin mold could destroy certain types of bacteria. Production increases allowed the price of penicillin to fall from about \$20 per dose in 1943 to \$0.55 per dose in 1946.

Source: inventors.about.com

a. Write a recursive formula for the number of bacteria alive before each application of the drug.

Let b_n represent the number of bacteria alive just before the nth application of the drug. 4.0×10^8 of these will be killed by the drug, leaving $b_n - 4.0 \times 10^8$. The number b_{n+1} of bacteria before the next application will have increased by 50%. So $b_{n+1} = 1.5(b_n - 4.0 \times 10^8)$, or $1.5b_n - 6.0 \times 10^8$.

b. Find the number of bacteria alive before the fifth application.

Before the first application of the drug, there were 1.0×10^9 bacteria alive, so $b_1 = 1.0 \times 10^9$.

$b_{n+1} = 1.5b_n - 6.0 \times 10^8$ Recursive formula

$b_{1+1} = 1.5b_1 - 6.0 \times 10^8$ $n = 1$

$b_2 = 1.5(1.0 \times 10^9) - 6.0 \times 10^8$ or 9.0×10^8

$b_{2+1} = 1.5b_2 - 6.0 \times 10^8$ $n = 2$

$b_3 = 1.5(9.0 \times 10^8) - 6.0 \times 10^8$ or 7.5×10^8

$b_{3+1} = 1.5b_3 - 6.0 \times 10^8$ $n = 3$

$b_4 = 1.5(7.5 \times 10^8) - 6.0 \times 10^8$ or 5.25×10^8

$b_{4+1} = 1.5b_4 - 6.0 \times 10^8$ $n = 4$

$b_5 = 1.5(5.25 \times 10^8) - 6.0 \times 10^8$ or 1.875×10^8

Before the fifth dose, there would be 1.875×10^8 bacteria alive.

CHECK Your Progress

A stronger dose of the drug can kill 6.0×10^8 bacteria.

2A. Write a recursive formula for the number of bacteria alive before each dose of the drug.

2B. How many of the stronger doses of the drug will kill all the bacteria?

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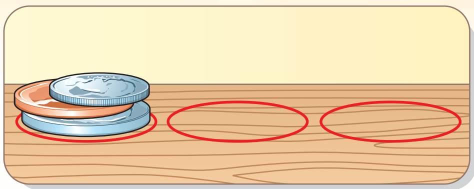
ALGEBRA LAB

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of n coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.



(continued on the next page)

MODEL AND ANALYZE

1. Draw three circles on a sheet of paper, as shown. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?
2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)
3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

MAKE A CONJECTURE

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of n different sized coins.

Study Tip

Look Back
To review the **composition of functions**, see Lesson 7-5.

Iteration **Iteration** is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or $f(f(x))$. If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or $f(f(f(x)))$, and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

EXAMPLE Iterate a Function

3 **Find the first three iterates x_1, x_2, and x_3 of the function $f(x) = 2x + 3$ for an initial value of $x_0 = 1$.**

$x_1 = f(x_0)$	Iterate the function.
$= f(1)$	$x_0 = 1$
$= 2(1) + 3$ or 5	Simplify.
$x_2 = f(x_1)$	Iterate the function.
$= f(5)$	$x_1 = 5$
$= 2(5) + 3$ or 13	Simplify.
$x_3 = f(x_2)$	Iterate the function.
$= f(13)$	$x_2 = 13$
$= 2(13) + 3$ or 29	Simplify.

The first three iterates are 5, 13, and 29.

CHECK Your Progress

3. Find the first four iterates, x_1, x_2, x_3, x_4, of the function $f(x) = x^2 - 2x - 1$ for an initial value of $x_0 = -1$.

CHECK Your Understanding

Example 1 (p. 658) **Find the first five terms of each sequence.**

1. $a_1 = 12, a_{n+1} = a_n - 3$
2. $a_1 = -3, a_{n+1} = a_n + n$
3. $a_1 = 0, a_{n+1} = -2a_n - 4$
4. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n$

Example 2
(p. 659)

BANKING **For Exercises 5 and 6, use the following information.**
Rita has deposited \$1000 in a bank account. At the end of each year, the bank posts 3% interest to her account, but then takes out a \$10 annual fee.

5. Let b_0 be the amount Rita deposited. Write a recursive equation for the balance b_n in her account at the end of n years.

6. Find the balance in the account after four years.

Example 3
(p. 660)

Find the first three iterates of each function for the given initial value.

7. $f(x) = 3x - 4, x_0 = 3$ **8.** $f(x) = -2x + 5, x_0 = 2$ **9.** $f(x) = x^2 + 2, x_0 = -1$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–17	1
18–21	3
22–27	2

Find the first five terms of each sequence.

10. $a_1 = -6, a_{n+1} = a_n + 3$

11. $a_1 = 13, a_{n+1} = a_n + 5$

12. $a_1 = 2, a_{n+1} = a_n - n$

13. $a_1 = 6, a_{n+1} = a_n + n + 3$

14. $a_1 = 9, a_{n+1} = 2a_n - 4$

15. $a_1 = 4, a_{n+1} = 3a_n - 6$

16. If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \geq 0$, find the value of a_5.

17. If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \geq 0$, then what is the value of a_4?

Find the first three iterates of each function for the given initial value.

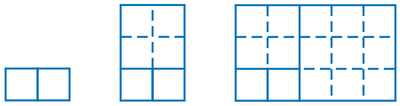
18. $f(x) = 9x - 2, x_0 = 2$

19. $f(x) = 4x - 3, x_0 = 2$

20. $f(x) = 3x + 5, x_0 = -4$

21. $f(x) = 5x + 1, x_0 = -1$

GEOMETRY **For Exercises 22–24, use the following information.**
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process.



Step 1 Step 2 Step 3

22. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with two original squares.

23. Write a recursive formula for the sequence of lengths added.

24. Identify the sequence in Exercise 23.

GEOMETRY **For Exercises 25–27, study the triangular numbers shown below.**

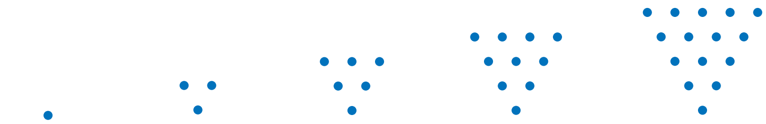


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

Real-World Career...
Loan Officer
Loan officers help customers through the loan application process. Their work may require frequent travel.

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25. Write a sequence of the first five triangular numbers.

26. Write a recursive formula for the nth triangular number t_n.

27. What is the 200th triangular number?

28. **LOANS** Miguel's monthly car payment is \$234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the \$10,000 loan after each of the first eight payments.

EXTRA PRACTICE
See pages 915, 936.

Math Online
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29. ECONOMICS If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a $70 MP3 player in four years if the rate of inflation remains constant.

Find the first three iterates of each function for the given initial value.

30. $f(x) = 2x^2 - 5, x_0 = -1$

31. $f(x) = 3x^2 - 4, x_0 = 1$

32. $f(x) = 2x^2 + 2x + 1, x_0 = \frac{1}{2}$

33. $f(x) = 3x^2 - 3x + 2, x_0 = \frac{1}{3}$

H.O.T. Problems

34. OPEN ENDED Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.

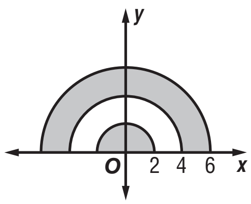
35. REASONING Is the statement $x_n \neq x_{n-1}$ *sometimes*, *always*, or *never* true if $x_n = f(x_{n-1})$? Explain.

36. CHALLENGE Are there a function $f(x)$ and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? Explain.

37. *Writing in Math* Use the information on page 658 to explain how the Fibonacci sequence is illustrated in nature. Include the 13th term in the sequence, with an explanation of what it tells you about the plant described.

STANDARDIZED TEST PRACTICE

38. ACT/SAT The figure is made of three concentric semicircles. What is the total area of the shaded regions?



A 4π units^2

B 10π units^2

C 12π units^2

D 20π units^2

39. REVIEW If x is a real number, for what values of x is the equation $\frac{4x - 16}{4} = x - 4$ true?

F all values of x

G some values of x

H no values of x

J impossible to determine

Spiral Review

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

40. $9 + 6 + 4 + \cdots$

41. $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$

42. $4 - \frac{8}{3} + \frac{16}{9} + \cdots$

Find the sum of each geometric series. (Lesson 11-4)

43. $2 - 10 + 50 - \cdots$ to 6 terms

44. $3 + 1 + \frac{1}{3} + \cdots$ to 7 terms

45. GEOMETRY The area of rectangle $ABCD$ is $6x^2 + 38x + 56$ square units. Its width is $2x + 8$ units. What is the length of the rectangle? (Lesson 6-3)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.**

46. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

47. $\frac{4 \cdot 3}{2 \cdot 1}$

48. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$